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Second quantization in a quon-algebra

Jacob Schach Møller

Aarhus University, Institute of Mathematics, Ny Munkegade, 8000 Aarhus C, Denmark

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Abstract. From the early days of quantum field theory it has been known that observables from quantum mechanics can be extended to observables in quantum fields: the so-called process of second quantization. The explicit form of the normal ordered expansion series for a second quantized observable is a quadratic form of the creation and annihilation operators. If we consider a quon-algebra described by the q-commutation relation $a(x)a^+(y)-qa^+(y)a(x)=\langle x,y\rangle I$, $-1\leqslant q\leqslant 1$, the normal ordered expansion series becomes quite complicated. For infinite statistics, i.e. for $q=0$, the expansion series is known. In the present paper we find the normal ordered expansion series for second-quantized, arbitrary observables.

1. Introduction

In recent literature several kinds of deformed canonical commutation relations are considered. They are often useful in the representation theory of quantum groups (Pusz and Woronowicz [1], [2], Biedenharn [3] and Macfarlane [4]). The deformations connected with the concept of quon-algebra seem to be quite useful (Greenberg [5] and Mohapatra [6]) and it is this type of deformations we are going to consider in the present paper. It is worth mentioning that quon-algebras are closely connected with the theory of q-series, e.g. the Gauss polynomials are identical with the q-binomial coefficients [7]. Quon-algebras are also related to non-canonical commutation relations in the first-order approximation, as proposed by Heisenberg [8]. Our paper, which is primarily inspired by Greenberg [5], is not concerned with the general theory of quon-algebras [9], but only with producing the normal ordered expansion series for the second quantization of observables, in particular the number operator. The treatment given in this paper is more general than the one used in [14] and [15].

2. Definitions and known results

The annihilation and creation operators acting on the (not completed) Fock space $\Gamma_0\mathscr{H}$ should obey the q-commutation relations

$$[a(x),a^+(y)]_q=\langle x,y\rangle I \tag{1}$$

where $[A,B]_q=AB-qBA$. The parameter q ranges from -1 to 1. The arguments x, y in the annihilation and creation operators are elements of the one-particle, real or complex, Hilbert space $\mathscr{H}$. We write ϕ for the vacuum and assume as usual that $a(x)\phi=0$. Given $x_1,x_2,\ldots,x_n\in\mathscr{H}$, we briefly write

$$x_1x_2\ldots x_n=a^+(x_1)a^+(x_2)\ldots a^+(x_n)\phi.$$

In what follows Σ_n denotes the permution group of n elements and $S_{n,m} \subset \Sigma_{n+m}$ denotes the subset of $n, m-n$ shuffles

$$S_{n.m} = \{\sigma \in \Sigma_{n+m} \mid \sigma(1) < \ldots < \sigma(n) \text{ and } \sigma(n+1) < \ldots < \sigma(n+m)\}.$$

For $\sigma \in \Sigma_n$ we let $\#\sigma$ denote the least number of transpositions of neighbouring elements needed to generate σ.

The operator $a(x)$ satisfies the q-Leibniz rule of derivation

$$a(x)(x_1 \ldots x_n) = \sum_{i=1}^{n} q^{-1}\langle x, x_i\rangle \cdot x_1 \ldots \hat{x}_i \ldots x_n. \tag{2}$$

The inner product between elements of the form $x_1, \ldots, x_n$ is

$$\langle x_1 \ldots x_n, y_1 \ldots y_m\rangle_q = \delta_{n,m} \cdot \det{}_q(\langle x_i, y_j\rangle) \tag{3}$$

where for an $n \times n$-matrix A we write

$$\det{}_q(A) = \sum_{i=1}^{n} q^{i-1} a_{1i} \cdot \det{}_q(A_{1i}) \qquad \det{}_q(a) = a.$$

This q-determinant can also be written as

$$\sum_{\sigma \in \Sigma_n} q^{\#\sigma} \cdot a_{1\sigma(1)} \ldots a_{n\sigma(n)}. \tag{4}$$

The inner product is positive definite [5, 10, 11]. The positive definiteness assures that for an orthonormal basis $\{e_n\}$, elements of the form $e_{j_{\sigma(1)}} \ldots e_{j_{\sigma(n)}}$ are linearly independent for $\sigma \in \Sigma_n / I(j_1, \ldots, j_n)$, where $I(j_1, \ldots, j_n)$ is the group of permutations that sends $j_1, \ldots, j_n$ into itself. There are exceptions for $q = \pm 1$, i.e. for the Fermi and the Bose algebras. In these algebras we additionally have the q-commutation relation $[a^+(x), a^+(y)]_q = 0$ which is not valid for other qs. This means that the $q = \pm 1$ algebras are considerably smaller than quon-algebras for $-1 < q < 1$.

In addition to the Leibniz rule (2), we use a result of [9] which provides a more general derivation formula. Namely, for $m \geqslant n$ we have

$$\begin{aligned} &a(x_n) \ldots a(x_1)(y_1 \ldots y_m) \\ &\quad = \sum_{\tau \in S_{n,m-n}} q^{\#\tau} \langle x_1 \ldots x_n, y_{\tau(1)} \ldots y_{\tau(n)}\rangle \cdot y_{\tau(n+1)} \ldots y_{\tau(m)}. \end{aligned} \tag{5}$$

This formula is essential for the proof of the main theorems of this paper.

3. Abstract second quantization

It is well known that a bounded operator U on the one-particle space extends to a homomorphism ΓU on the Fock-space (cf [12] for a proof which can easily be generalized to arbitrary q). This we use to construct the second quantization of an observable A. We extend the unitary semigroup e^{itA} generated by A to a unitary semigroup Γe^{itA} on the Fock-space. The generator of Γe^{itA} constitutes the second quantization, $d\Gamma A$, of A. We have

$$\begin{aligned} d\Gamma A(f \cdot g) &= -i \cdot \frac{d}{dt}[\Gamma e^{itA}(f \cdot g)]|_{t=0} \\ &= (d\Gamma A f) \cdot g + f \cdot (d\Gamma A g) \end{aligned} \tag{6}$$

which means that $d\Gamma A$ is a derivation. Furthermore, since $\Gamma\, e^{itA}$ extends e^{itA}, we have that for $x \in \mathcal{H} \subset \Gamma_0\mathcal{H}$

$$d\Gamma A(x) = Ax. \tag{7}$$

Notice that these two properties uniquely determine $d\Gamma A$.

4. Normal ordered expansion series

In the case of Bose and Fermi algebras the explicit normal ordered expansion series of $d\Gamma A$ is a quadratic form of annihilation and creation operators, i.e. for $q = \pm 1$

$$d\Gamma A = \sum_{n=1}^{\infty} a^{+}(Ae_n)a(e_n) \tag{8}$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$. This definition is independent of the choice of basis.

For $q = 0$ the number operator, N [5], is given by

$$N = d\Gamma I = \sum_{n=1}^{\infty} \sum_{k_1,\ldots,k_n} e_{k_1} \ldots e_{k_n} a(e_{k_n}) \ldots a(e_{k_1}). \tag{9}$$

From now on we will require that $-1 < q < 1$. Before giving the form of the normal ordered expansion series of $d\Gamma A$ for an arbitrary linear operator A, we consider the case of $A = I$, where I denotes the identity.

Theorem 1. Let $\Gamma_0\mathcal{H}$ be a quon-algebra generated by a one-particle Hilbert space $\mathcal{H}$. The number operator N, i.e. the operator which has all products of the form $x_1 \ldots x_n$, $x_i \in \mathcal{H}$, an eigenvectors corresponding to the eigenvalue n, has the normal ordered expansion series

$$N = \sum_{n=1}^{\infty} \sum_{k_1,\ldots,k_n} \sum_{\sigma\in\Sigma_n} \alpha_n^{\sigma} \cdot e_{k_1} \ldots e_{k_n} \cdot a(e_{k_{\sigma(n)}}) \ldots a(e_{k_{\sigma(1)}})$$

where the coefficients are given by the recursive $(m! \times m!)$ matrix equation

$$\begin{pmatrix} 1 & \cdots & q^{\#\sigma} & \cdots \\ \vdots & \ddots & & \\ q^{\#s} & & q^{\#\sigma s} & \\ \vdots & & & \ddots \end{pmatrix} \begin{pmatrix} \alpha_m^{id} \\ \vdots \\ \alpha_m^{\sigma} \\ \vdots \end{pmatrix} = \begin{pmatrix} m \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} - \begin{pmatrix} R_{m,id} \\ \vdots \\ R_{m,s} \\ \vdots \end{pmatrix}$$

with

$$R_{m,s} = \sum_{k=n_s}^{m-1} \sum_{\tau\in\Sigma_k} \alpha_k^{\tau} q^{\#\tau s} \tag{10}$$

$$n_s = \min\{1 \leq k \leq m-1 \mid s(k+1) < \ldots < s(m)\}$$

and

$$\alpha_1^{id} = 1.$$

Proof. First we compute $N(x_1 \ldots x_m)$, where $x_i \in \mathscr{H}$.

$$\begin{aligned}
N(x_1 \ldots x_m) &= \sum_{n=1}^{\infty} \sum_{k_1,\ldots,k_n} \sum_{\sigma\in\Sigma_n} \alpha_n^\sigma e_{k_1} \ldots e_{k_n} a(e_{k_\sigma(n)}) \ldots a(e_{k_{\sigma(1)}})(x_1 \ldots x_m) \\
&= \sum_{n=1}^{m} \sum_{k_1,\ldots,k_n} \sum_{\sigma\in\Sigma_n} \sum_{\kappa\in S_{n,m-n}} \alpha_n^\sigma q^{\#\kappa} \\
&\quad \times e_{k_1} \ldots e_{k_n} \langle e_{k_{\sigma(1)}} \ldots e_{k_{\sigma(n)}}, x_{\kappa(1)} \ldots x_{\kappa(n)} \rangle x_{\kappa(n+1)} \ldots x_{\kappa(m)} \\
&= \sum_{n=1}^{m} \sum_{k_1,\ldots,k_n} \sum_{\tau,\sigma\in\Sigma_n} \sum_{\kappa\in S_{n,m-n}} \alpha_n^\sigma q^{\#\kappa+\#\tau} \\
&\quad \times e_{k_1} \ldots e_{k_n} \langle e_{k_{\sigma(1)}}, x_{\tau\kappa(1)} \rangle \ldots \langle e_{k_{\sigma(n)}}, x_{\tau\kappa(n)} \rangle x_{\kappa(n+1)} \ldots x_{\kappa(m)} \\
&= \sum_{n=1}^{m} \sum_{\tau,\sigma\in\Sigma_n} \sum_{\kappa\in S_{n,m-n}} \alpha_n^\sigma q^{\#\tau\kappa} x_{\sigma^{-1}\tau\kappa(1)} \ldots x_{\sigma^{-1}\tau\kappa(n)} x_{\kappa(n+1)} \ldots x_{\kappa(m)}. \qquad (11)
\end{aligned}$$

These computations use (3), (4) and (5). To find the coefficient of $x_{s(1)} \ldots x_{s(m)}$, where $s \in \Sigma_m$, we proceed as follows. Define

$$n_s = \min\{1 \leqslant k \leqslant m-1 \mid s(k+1) < \ldots < s(m)\}.$$

For a term in the summation over $S_{n,m-n}$ in (11) to contribute to a coefficient corresponding to a permutation s, the shuffle must satisfy the condition

$$\kappa(n+1) = s(n+1), \ldots, \kappa(m) = s(m) \quad \text{and} \quad \kappa \in S_{n,m-n}.$$

Therefore the summation over $S_{n,m-n}$ contributes with only one term. The $\kappa \in S_{n,m-n}$ that gives the contribution is uniquely determined by s, and hence we will denote it by κ_s.

We can now write the coefficient of $x_{s(1)} \ldots x_{s(m)}$ as

$$\sum_{n=n_s}^{m} \sum_{\substack{\tau,\sigma\in\Sigma_n \\ \sigma^{-1}\tau\kappa_s = s}} \alpha_n^\sigma q^{\#\tau\kappa_s} = \sum_{n=n_s}^{m} \sum_{\sigma\in\Sigma_n} \alpha_n^\sigma q^{\#\sigma s}$$

which yields the recursive $(m! \times m!)$ matrix equation as stated above.

We verify that $\alpha_1^{id} = 1$ by applying the normal ordered expansion series for the number operator to a single element x and requiring the eigenvalue to be 1.

The uniqueness of α_m^σ follows by noticing that the matrix to be inverted is actually unitarily equivalent to the matrix

$$(\langle e_{\sigma(1)} \ldots e_{\sigma(m)}, e_{s(1)} \ldots e_{s(m)} \rangle)$$

where both σ and s run through Σ_m, and $e_1, \ldots, e_m$ are elements from an orthonormal basis of $\mathscr{H}$. Since the inner product (3) is strictly positive definite [14], the determinant of this latter matrix is different from zero (or consult [5] for an explicit expression for this determinant). □

It is, however, inconvenient to carry out calculations with a recursive matrix equation. The next proposition removes the recursive structure and gives an explicit matrix equation for α_n^σ. We start with a few notational comments and a combinatorial result.

Theorem 2. Let $\sigma \in \Sigma_n$. The permutation σ can be decomposed into a composition of n, m shuffles $\kappa_1, \ldots, \kappa_{k_\sigma}$ such that $\kappa_i \in S_{n_i, n_{i-1}-n_i} \subset \Sigma_{n_{i-1}}$ and $n_0 = n$, where Σ_{n_i} are embedded canonically into Σ_n. This decomposition is unique if k_σ is minimized. For all such decompositions the following homomorphism holds

$$\#\sigma = \#\kappa_{k_\sigma} + \ldots + \#\kappa_1. \qquad \square$$

From now on k_σ denotes the smallest number of shuffles $\kappa_1, \ldots, \kappa_{k_\sigma}$, $k_i \in S_{n_i, n_{i-1}-n_i}$, needed to decompose σ. Notice that in this context the number n_σ used in the formula for the coefficients is equal to n_1.

Now we are ready to state the non-recursive matrix equations for the α_n^σ.

Theorem 3. The following matrix equations determine the coefficients in the normal ordered series for $N = \mathrm{d}\Gamma I$

$$\begin{pmatrix} 1 & \cdots & q^{\#\sigma} \cdots \\ \vdots & \ddots & \\ q^{\#s} & & q^{\#\sigma s} \\ \vdots & & & \ddots \end{pmatrix} \cdot \begin{pmatrix} \alpha_n^{id} \\ \vdots \\ \alpha_n^\sigma \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ (-1)^{k_s} \cdot \delta_{n_{k_s}, n-k_s} \cdot q^{\#s} \\ \vdots \end{pmatrix}$$

where the vectors and the matrix are indexed by permutations from Σ_n.

Proof. The proof consists of a series of technical lemmas, where we compute the right-hand side of the recursive matrix equation.

The idea is to evaluate the factors $R_{m,\sigma}$ by recursion using a decomposition into shuffles. First we calculate $R_{m,\kappa}$, where κ is a shuffle.

Lemma 1. For $n \geqslant 1$ we have the identity

$$\sum_{\sigma \in \Sigma_n} \alpha_n^\sigma q^{\#\sigma} = 1$$

Proof. We have

$$\sum_{\sigma \in \Sigma_n} \alpha_n^\sigma q^{\#\sigma} = \left\langle \begin{pmatrix} \alpha_n^{id} \\ \vdots \\ \alpha_n^\sigma \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 \\ \vdots \\ q^{\#\sigma} \\ \vdots \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} \alpha_n^{id} \\ \vdots \\ \alpha_n^\sigma \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 & \cdots & q^{\#\sigma} & \cdots \\ \vdots & \ddots & & \\ q^{\#s} & & q^{\#\sigma s} & \\ \vdots & & & \ddots \end{pmatrix}^t \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} \right\rangle.$$

$$= n - R_{n,id}.$$

Since $n_{id} = 1$, we conclude by using downward induction that the lemma holds. $\square$

Corollary 1. For $id \in \Sigma_n$ we have

$$R_{n,id} = n - 1. \qquad \square$$

Definition. Let $z:\mathbb{N}\to\mathbb{Z}$ be defined by the series

$$z(a)=\sum_{\substack{k_1,\ldots,k_j\geq 1\\ \Sigma k_i=a}}(-1)^{j+1}.$$

Lemma 2. Let $\kappa\in S_{n_\kappa,m-n_\kappa}\backslash\{id\}\subset\Sigma_m$. We have

$$R_{m,\kappa}=z(m-n_\kappa)\cdot q^{\#\kappa}.$$

Proof. Recalling the definition (10) of $R_{m,s}$, we see that given n, $m-1\geq n>n_\kappa$, we can decompose κ uniquely into two shuffles κ_n and τ_n such that

$$\kappa_n\in S_{n,m-n}\qquad\qquad \tau_n\in S_{n_\kappa,n-n_\kappa}\backslash\{id\}$$

and

$$\#\kappa=\#\kappa_n+\#\tau_n.$$

For $n=n_\kappa$, τ_n is the identity, and thus the inner sum yields $q^{\#\kappa_n}=q^{\#\kappa}$. We can now repeat this process on the other summation terms by utilizing the recursive matrix equation until the sum contains only the term indexed by n_κ. This procedure is repeated as many times as the number of partitions of $m-n_\kappa$. The factor $(-1)^{j+1}$ comes from using the matrix equation a number of times equalling the partition number minus one. □

Lemma 3. Let $s\in\Sigma_m\backslash\{id\}$ such that $\sigma=\kappa\circ\tau$, where $\kappa\in S_{n_s,m-n_s}$ and $\tau\in\Sigma_n\backslash\{id\}\subset\Sigma_m$ canonically. We have

$$R_{m,s}=-z(m-n_s)\cdot q^{\#\kappa}\cdot R_{n_s,\tau}.$$

Proof. We use exactly the same procedure as in the proof of the last lemma until we reach the stage with one term in the sum. At this stage we do not end up with a sum that yields one. We now use the recursive matrix equation once more which gives the sign of $R_{m,s}$. □

Corollary 2. Let $s\in\Sigma_m$ and $\kappa_1,\ldots,\kappa_{k_s}$ be the unique decomposition of s into shuffles. We have

$$R_{m,s}=(-1)^{k_s+1}\cdot\left(\prod_{i=1}^{k_s}z(n_{i-1}-n_i)\right)\cdot q^{\#s}. \qquad □$$

Lemma 4. We have the simple result

$$z(a)=\delta_{a,1}.$$

Proof. The sum in the defining series goes through all partitions of a. By simple combinatorics we find that there are

$$\binom{a-1}{j-1}$$

partitions with partition number j. This fact combined with a suitable combinatorial identity yields the result. □

Corollary 3. For $s \in \Sigma_m \backslash \{id\}$ we have

$$R_{m,s} = \begin{cases} (-1)^{k_s+1} \cdot q^{\#s} & \kappa_i \in S_{m-i,1} \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

By this we have obtained the non-recursive matrix equation.

For $q=0$ the equation for α_n^σ is easily solved and the result is

$$\alpha_n^\sigma = \delta_{\sigma,id}$$

in agreement with (9).

For q equal to zero the generalization from $d\Gamma I$ to $d\Gamma A$ is easy. For a self-adjoint operator A on $\mathscr{H}$ we have

$$d\Gamma A = \sum_{n=1}^{\infty} \sum_{k_1,\dots,k_n} \sum_{\sigma\in\Sigma_n} \alpha_n^\sigma e_{k_1} \dots e_{k_{n-1}} A e_{k_n} a(e_{k_{\sigma(n)}}) \dots a(e_{k_{\sigma(1)}}) \tag{12}$$

It is easily verified that this is the desired normal ordered expansion series for $d\Gamma A$ in the case $q=0$.

Calculations analogous to the ones used above show that for general q we have

$$\begin{aligned} &d\Gamma A(x_1 \dots x_n) \\ &\quad = \sum_{i=1}^{n} x_1 \dots Ax_i \dots x_n + \sum_{\sigma\in\Sigma_n\backslash id} (-1)^{k_\sigma} \cdot q^{\#\sigma} \\ &\qquad \times (x_{\sigma(1)} \dots Ax_{\sigma(n_\sigma)} \dots x_{\sigma(n)} - x_{\sigma(1)} \dots Ax_{\sigma(n_\sigma+1)} \dots x_{\sigma(n)}). \end{aligned}$$

However, for $q \neq 0$ this is not a derivation and hence not of interest. We will consider a more general normal ordered expansion series. The computations are to a large extend the same as for the number operator so that we can utilize practically all the previous results.

Theorem 4. The normal ordered expansion series for the second quantization $d\Gamma A$, of $A: \mathscr{H} \to \mathscr{H}$, a self-adjoint/anti-self-adjoint operator is

$$d\Gamma A = \sum_{n=1}^{\infty} \sum_{j=1}^{n} \sum_{k_1,\dots,k_n} \sum_{\sigma\in\Sigma_n} \alpha_{n,j}^\sigma e_{k_1} \dots Ae_{k_j} \dots e_{k_n} a(e_{k_{\sigma(n)}}) \dots a(e_{k_{\sigma(1)}})$$

where the coefficients are given by the matrix equations

$$\begin{pmatrix} 1 & \cdots & q^{\#\sigma} & \cdots \\ \vdots & \ddots & & \\ q^{\#s} & & q^{\#\sigma s} & \\ \vdots & & & \ddots \end{pmatrix} \cdot \begin{pmatrix} \alpha_{n,j}^{id} \\ \vdots \\ \alpha_{n,j}^{\sigma} \\ \vdots \end{pmatrix} = \begin{pmatrix} \delta_{n,j} \\ \vdots \\ \delta_{j,n_{k_s}} \cdot \delta_{n_{k_s},n-k_s} (-1)^{k_s} \cdot q^{\#s} \\ \vdots \end{pmatrix}$$

for $j \leq n$, and for $j > n$ we define $\alpha_{n,j}^\sigma = 0$.

Proof. First we calculate a recursive matrix equation similar to the one for the number operator. To utilize the calculations carried out for N we must exchange the outer sums in the series for $d\Gamma A$. Compared to the case of the number operator, there are m-times as many coefficients to find, namely coefficients to the terms

$$x_{s(1)} \dots Ax_{s(j)} \dots x_{s(m)}.$$

This gives $m-1$ recursive equations and one non-recursive equation, namely

$$\begin{pmatrix} 1 & \cdots & q^{*\sigma} & \cdots \\ \vdots & \ddots & & \\ q^{*s} & & q^{*\sigma s} & \\ \vdots & & & \ddots \end{pmatrix}\begin{pmatrix} \alpha^{id}_{m,j} \\ \vdots \\ \alpha^{\sigma}_{m,j} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} - \begin{pmatrix} R^{id}_{m,j} \\ \vdots \\ R^{s}_{m,j} \\ \vdots \end{pmatrix}$$

where

$$R^{s}_{m,j} = \sum_{n=\max(n_s,j)}^{m-1} \sum_{\sigma\in\Sigma_n} \alpha^{\sigma}_{n,j}\cdot q^{*\sigma s} \qquad j<m$$
$$R^{s}_{m,j} = 0 \qquad j\geqslant m$$

and

$$\alpha^{id}_{1,1} = 1.$$

Now the procedure will be similar to the one used for the number operator. We being with the following:

Lemma 5. For $n\geqslant 1$ we have

(i) $$\sum_{\sigma\in\Sigma_n} \alpha^{\sigma}_{n,j}\cdot q^{*\sigma} = \delta_{n,j}$$

and for $s\in\Sigma_n$ we have

(ii) $$\sum_{\sigma\in\Sigma_n} \alpha^{\sigma}_{n,n}\cdot q^{*\sigma s} = \delta_{s,id}.$$

Proof. The second part follows directly from the recursive matrix equations (which in this case are actually not recursive). To prove (i) we calculate

$$\sum_{\sigma\in\Sigma_m} \alpha^{\sigma}_{m,j}\cdot q^{*\sigma} = 1 - \sum_{n=j}^{m-1}\sum_{\sigma\in\Sigma_n} \alpha^{\sigma}_{n,j}\cdot q^{*\sigma}$$
$$= -\sum_{n=j+1}^{m-1}\sum_{\sigma\in\Sigma_n} \alpha^{\sigma}_{n,j}\cdot q^{*\sigma}$$

which by induction is equal to $\delta_{m,j}$. □

Corollary 4. For $id\in\Sigma_n$ we have

$$R^{id}_{m,j} = 1-\delta_{m,j}.$$ □

Lemma 6. Let $\kappa\in S_{k,m-k}\backslash\{id\}$. Then

$$R^{\kappa}_{m,j} = \delta_{k,j}\cdot\delta_{j,m-1}\cdot q^{*\kappa}.$$

Proof. First consider the case $j\leqslant n_\kappa = k$. We have

$$R^{\kappa}_{m,j} = \sum_{n=k}^{m-1}\sum_{\sigma\in\Sigma_n} \alpha^{\sigma}_{n,j}\cdot q^{*\sigma\kappa}.$$

As in the proof of the analogous lemma concerning the number operator, we have

$$R^{\kappa}_{m,j} = z(m-k)\cdot q^{*\kappa}\cdot \sum_{\sigma\in\Sigma_k} \alpha^{\sigma}_{k,j}\cdot q^{*\sigma}$$

which yields the result for $j\leqslant k$. For $j>k$ we go through a similar procedure but then we end up with a sum as in part (ii) of lemma 5, which concludes the proof. □

Lemma 7. Write $s \in \Sigma_m \backslash \{id\}$ in the form $s = \kappa \circ \tau$, where $\kappa \in S_{n_s, m-n_s}$ and $\tau \in \Sigma_n \backslash \{id\}$. Then for $j < m$ we have the identity

$$R^s_{m,j} = -\delta_{n_s, m-1} \cdot q^{\# \kappa} \cdot R^{\tau}_{n_s, j}.$$

Proof. The proof is the same as in the case of the number operator. □

Now the theorem is a direct consequence of lemma 7.

For $q = 0$ the equations are easily solved and they yield

$$\alpha^{\sigma}_{n,j} = \delta_{\sigma, id} \cdot \delta_{n,j}$$

in agreement with (12).

5. Concluding remarks

Due to the obvious correlation between the coefficients of N and $\mathrm{d}\Gamma A$ we get the following:

Corollary 5. Let α^{σ}_n be the coefficients in the normal ordered expansion for N, and let $\alpha^{\sigma}_{n,j}$ be the coefficients in the normal ordered expansion for $\mathrm{d}\Gamma A$. We have

$$\alpha^{\sigma}_n = \sum_{j=1}^{n} \alpha^{\sigma}_{n,j}.$$ □

To understand the limit behaviour of the coefficients of $\mathrm{d}\Gamma A$ and N when $|q|$ tends to 1, consider an element e of an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ in $\mathscr{H}$. We compute

$$N(e^m) = \sum_{n=1}^{m} \left(\sum_{\sigma \in \Sigma_n} \alpha^{\sigma}_n \right) e^n \cdot a(e)^n (e^m)$$

$$= \sum_{n=1}^{m} \mathscr{A}^+_n \frac{m^q\,!}{(m-n)^q\,!} \cdot e^m$$

where

$$\mathscr{A}^+_m = \sum_{\sigma \in \Sigma_m} \alpha^{\sigma}_m.$$

This gives a recursive formula for $\mathscr{A}^+_m$.

$$\mathscr{A}^+_m = \frac{m}{m^q\,!} - \sum_{n=1}^{m-1} \frac{\mathscr{A}^+_n}{(m-n)^q\,!}.$$

Notice that we have as well

$$\mathscr{A}^+_m = \sum_{j=1}^{m} \sum_{\sigma \in \Sigma_m} \alpha^{\sigma}_{m,j}$$

so that $\mathscr{A}^+_m$ also reflects the behaviour of $\alpha^{\sigma}_{m,j}$. Since $\mathscr{A}^+_2 = (1-q)/(1+q)$, $\mathscr{A}^+_1 = 1$ and $n^q\,! \to n!$ for $q \to 1$, easy induction shows that if $m > 1$, we have $\mathscr{A}^+_m \to 0$ for $q \to 1$. This result is quite natural since the number operator converges strongly to the Bose number operator as q tends to 1, and in the Bose case the terms $a(e_{\sigma(n)})\ldots a(e_{\sigma(1)})$ are identical. Following this line of reasoning, we define

$$\mathscr{A}_m^- = \sum_{\sigma\in\Sigma_m} (-1)^{\#\sigma}\cdot\alpha_m^\sigma$$

expecting that $\mathscr{A}_m^- \to 0$ for $q\to -1$, $m>1$. This holds at least for $m=2$.

We conclude by presenting a few explicitly computed coefficients

$$\alpha_1^{id}=1 \qquad \alpha_2^{id}=\frac{1+q^2}{1-q^2} \qquad \alpha_2^{(2,1)}=\frac{-2q}{1-q^2}$$

$$\alpha_{1,1}^{id}=1 \qquad \alpha_{2,1}^{id}=\frac{q^2}{1-q^2} \qquad \alpha_{2,1}^{(2,1)}=\frac{-q}{1-q^2}$$

$$\alpha_{2,2}^{id}=\frac{1}{1-q^2} \qquad \alpha_{2,2}^{(2,1)}=\frac{-q}{1-q^2} \qquad \mathscr{A}_1^+=1$$

$$\mathscr{A}_2^+=\frac{1-q}{1+q} \qquad \mathscr{A}_3^+=\frac{(1-q)^3}{(1+q)(1-q+q^2)}$$

$$\mathscr{A}_1^{-1}=1 \qquad \mathscr{A}_2^-=\frac{1+q}{1-q}.$$

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